arXiv:math/0409599v3 [math.QA] 1 Apr 2005

YETTER-DRINFELD MODULES OVER WEAK BIALGEBRAS

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Abstract. We discuss properties of Yetter-Drinfeld modules over weak bialgebras over commutative rings. The categories of left-left, left-right, right-left and right-right Yetter-Drinfeld modules over a weak Hopf algebra are isomorphic as braided monoidal categories. Yetter-Drinfeld modules can be viewed as weak Doi-Hopf modules, and, a fortiori, as weak entwined modules. If H is finitely generated and projective, then we introduce the Drinfeld double using duality results between entwining structures and smash product structures, and show that the category of Yetter-Drinfeld modules is isomorphic to the category of modules over the Drinfeld double. The category of finitely generated projective Yetter-Drinfeld modules over a weak Hopf algebra has duality.

Introduction

Weak bialgebras and Hopf algebras are generalizations of ordinary bialgebras and Hopf algebras in the following sense: the defining axioms are the same, but the multiplicativity of the counit and comultiplicativity of the unit are replaced by weaker axioms. The easiest example of a weak Hopf algebra is a groupoid algebra; other examples are face algebras [10], quantum groupoids [19], generalized Kac algebras [25] and quantum transformation groupoids [18]. Temperley-Lieb algebras give rise to weak Hopf algebras (see [18]). A purely algebraic study of weak Hopf algebras has been presented in [2]. A survey of weak Hopf algebras and their applications may be found in [18]. It has turned out that many results of classical Hopf algebra theory can be generalized to weak Hopf algebras.

Yetter-Drinfeld modules over finite dimensional weak Hopf algebras over fields have been introduced by Nenciu [16]. It is shown in [16] that the category of finite dimensional Yetter-Drinfeld modules is isomorphic to the category of finite dimensional modules over the Drinfeld double, as introduced in the appendix of [1]. It is also shown that this category is braided isomorphic to the center of the category of finite dimensional H-modules. In this note, we discuss Yetter-Drinfeld modules over weak bialgebras over commutative rings. The results in [16] are slightly generalized and more properties are given.

In Section 2, we compute the weak center of the category of modules over a weak bialgebra H, and show that it is isomorphic to the category of Yetter-Drinfeld modules. If H is a weak Hopf algebra, then the weak center equals the center. In this situation, properties of the center construction can be applied to show that the four categories of Yetter-Drinfeld modules, namely the left-left, left-right, right-left and

1991 *Mathematics Subject Classification.* 16W30.

Research supported by the projects G.0278.01 "Construction and applications of non-commutative geometry: from algebra to physics" from FWO-Vlaanderen and "New computational, geometric and algebraic methods applied to quantum groups and differential operators" from the Flemish and Chinese governments.

right-right versions, are isomorphic as braided monoidal categories. Here we apply methods that have been used before in [5], in the case of quasi-Hopf algebras.
In [7], it was observed that Yetter-Drinfeld modules over a classical Hopf algebra are special cases of Doi-Hopf modules, as introduced by Doi and Koppinen (see [8, 13]). In Section 3, we will show that Yetter-Drinfeld modules over weak Hopf algebras are weak Doi-Hopf modules, in the sense of Böhm [1], and, a fortiori, weak entwined modules [6], and comodules over a coring [4].
The advantage of this approach is that it leads easily to a new description of the Drinfeld double of a finitely generated projective weak Hopf algebra, using methods developed in [6]: we define the Drinfeld double as a weak smash product of H and its dual. We show that our Drinfeld double is equal to the Drinfeld double of [1, 16] (see Proposition 4.3) and anti-isomorphic to the Drinfeld double of [17] (see Proposition 4.5). In Section 5, we show that the category of finitely generated projective Yetter-Drinfeld modules over a weak Hopf algebra has duality.
In Sections 1.1 and 1.2, we recall some general properties of weak bialgebras and Hopf algebras. Further detail can be found in [4, 2, 18]. In Section 1.3, we recall the center construction, and in Section 1.4, we recall the notions of weak Doi-Hopf modules, weak entwining structures and weak smash products.

1. Preliminary results

1.1. **Weak bialgebras.** Let k be a commutative ring. Recall that a weak k-bialgebra is a k-module with a k-algebra structure (μ,η) and a k-coalgebra structure (Δ,ε) such that $\Delta(hk)=\Delta(h)\Delta(k)$, for all $h,k\in H$, and

$$\Delta^2(1) = 1_{(1)}\otimes 1_{(2)}1_{(1')}\otimes 1_{(2')} = 1_{(1)}\otimes 1_{(1')}1_{(2)}\otimes 1_{(2')}, \tag{1}$$

$$\varepsilon(hkl) = \varepsilon(hk_{(1)})\varepsilon(k_{(2)}l) = \varepsilon(hk_{(2)})\varepsilon(k_{(1)}l), \tag{2}$$

for all $h,k,l\in H$. We use the Sweedler-Heyneman notation for the comultiplication, namely

$$\Delta(h) = h_{(1)}\otimes h_{(2)} = h_{(1')}\otimes h_{(2')}.$$

We summarize the elementary properties of weak bialgebras. The proofs are direct applications of the defining axioms (see [2, 18]). We have idempotent maps $\varepsilon_t,\ \varepsilon_s: H\to H$ defined by

$$\varepsilon_t(h)=\varepsilon(1_{(1)}h)1_{(2)}\ ;\varepsilon_s(h)=1_{(1)}\varepsilon(h1_{(2)}).$$

ε_t and ε_s are called the target map and the source map, and their images $H_t=\mathrm{Im}\,(\varepsilon_t)=\mathrm{Ker}\,(H-\varepsilon_t)$ and $H_s=\mathrm{Im}\,(\varepsilon_s)=\mathrm{Ker}\,(H-\varepsilon_s)$ are called the target and source space. For all $g,h\in H$, we have

$$h_{(1)}\otimes\varepsilon_t(h_{(2)}) = 1_{(1)}h\otimes 1_{(2)} \text{ and } \varepsilon_s(h_{(1)})\otimes h_{(2)} = 1_{(1)}\otimes h1_{(2)}, \tag{3}$$

and

$$h\varepsilon_t(g)=\varepsilon(h_{(1)}g)h_{(2)} \text{ and } \varepsilon_s(g)h=h_{(1)}\varepsilon(gh_{(2)}). \tag{4}$$

From (4), it follows immediately that

$$\varepsilon(h\varepsilon_t(g))=\varepsilon(hg) \text{ and } \varepsilon(\varepsilon_s(g)h)=\varepsilon(gh). \tag{5}$$

The source and target space can be described as follows:

$$H_t=\{h\in H\mid \Delta(h)=1_{(1)}h\otimes 1_{(2)}\}=\{\phi(1_{(1)})1_{(2)}\mid \phi\in H^*\}; \tag{6}$$

$$H_s=\{h\in H\mid \Delta(h)=1_{(1)}\otimes h1_{(2)}\}=\{1_{(1)}\phi(1_{(2)})\mid \phi\in H^*\}. \tag{7}$$

We also have

$$\varepsilon_t(h)\varepsilon_s(k) = \varepsilon_s(k)\varepsilon_t(h), \tag{8}$$

and its dual property

$$\varepsilon_s(h_{(1)}) \otimes \varepsilon_t(h_{(2)}) = \varepsilon_s(h_{(2)}) \otimes \varepsilon_t(h_{(1)}). \tag{9}$$

Finally $\varepsilon_s(1) = \varepsilon_t(1) = 1$, and

$$\varepsilon_t(h)\varepsilon_t(g) = \varepsilon_t(\varepsilon_t(h)g) \text{ and } \varepsilon_s(h)\varepsilon_s(g) = \varepsilon_s(h\varepsilon_s(g)). \tag{10}$$

This implies that H_s and H_t are subalgebras of H.

Lemma 1.1. *Let H be a weak bialgebra over a commutative ring. Then $\Delta(1) \in H_s \otimes H_t$.*

Proof. Applying $H\otimes\varepsilon\otimes H$ to (1), we find that $1_{(1)}\otimes 1_{(2)} = \varepsilon_s(1_{(1)})\otimes 1_{(2)} \in H_s\otimes H$ and $1_{(1)} \otimes 1_{(2)} = 1_{(1)} \otimes \varepsilon_t(1_{(2)}) \in H \otimes H_t$. Now let $K_s = \mathrm{Ker}\,(\varepsilon_s)$, $K_t = \mathrm{Ker}\,(\varepsilon_t)$. Then $H = H_s \oplus K_s = H_t \oplus K_t$, and

$$H \otimes H = H_s \otimes H_t \oplus H_s \otimes K_t \oplus K_s \otimes H_t \oplus K_s \otimes K_t,$$

so it follows that $H_s \otimes H_t = H \otimes H_t \cap H_s \otimes H$. □

The target and source map for the weak bialgebra H^{op} are

$$\overline{\varepsilon}_t(h) = \varepsilon(h1_{(1)})1_{(2)} \in H_t \text{ and } \overline{\varepsilon}_s(h) = \varepsilon(1_{(2)}h)1_{(1)} \in H_s. \tag{11}$$

$\overline{\varepsilon}_t$ and $\overline{\varepsilon}_s$ are also projections.

The source and target space are anti-isomorphic, and they are separable Frobenius algebras over k. This was first proved for weak Hopf algebras (see [2]), and then generalized to weak bialgebras (see [22]).

Lemma 1.2. [22] *Let H be a weak bialgebra. Then $\overline{\varepsilon}_s$ restricts to an anti-algebra isomorphism $H_t \to H_s$ with inverse ε_t, and $\overline{\varepsilon}_t$ restricts to an anti-algebra isomorphism $H_s \to H_t$ with inverse ε_s.*

Proposition 1.3. [22] *Let H be a weak bialgebra. Then H_s and H_t are Frobenius separable k-algebras. The separability idempotents of H_t and H_s are*

$$e_t = \varepsilon_t(1_{(1)}) \otimes 1_{(2)} = 1_{(2)} \otimes \overline{\varepsilon}_t(1_{(1)});$$

$$e_s = 1_{(1)} \otimes \varepsilon_s(1_{(2)}) = \overline{\varepsilon}_s(1_{(2)}) \otimes 1_{(1)}.$$

The Frobenius systems for H_t and H_s are respectively $(e_t, \varepsilon_{|H_t})$ and $(e_s, \varepsilon_{|H_s})$. In particular, we have for all $z \in H_t$ that

$$z\varepsilon_t(1_{(1)}) \otimes 1_{(2)} = \varepsilon_t(1_{(1)}) \otimes 1_{(2)}z. \tag{12}$$

It was shown in [17] that the category of modules over a weak Hopf algebra is monoidal; it follows from the results of [22] that this property can be generalized to weak bialgebras. We explain now how this can be done directly.
Let M be a left H-module. By restriction of scalars, M is a left H_t-module; M becomes an H_t-bimodule, if we define a right H_t-action by

$$m \cdot z = \overline{\varepsilon}_s(z)m.$$

Let $M, N \in {}_H\mathcal{M}$, the category of left H-modules. We define

$$M \otimes_t N = \Delta(1)(M \otimes N),$$

the k-submodule of $M \otimes N$ generated by elements of the form $1_{(1)} \otimes 1_{(2)}$. $M \otimes_t N$ is a left H-module, with left diagonal action $h \cdot (m \otimes n) = h_{(1)}m \otimes h_{(2)}n$. It follows from (1) that the tensor product $\otimes_t$ is associative. Observe that

$$M \otimes_t N \otimes_t P = \Delta^2(1)(M \otimes N \otimes P).$$

$H_t \in {}_H\mathcal{M}$, with left H-action $h \rightharpoonup z = \varepsilon_t(hz)$. The induced H_t-bimodule structure is given by left and right multiplication by elements of H_t.
For $M, N \in {}_H\mathcal{M}$, consider the projection

$$\pi: \ M \otimes N \to M \otimes_t N, \ \pi(m \otimes n) = 1_{(1)}m \otimes 1_{(2)}n.$$

Applying $\overline{\varepsilon}_s \otimes H_t$ to (12), we find

$$\overline{\varepsilon}_s(z\varepsilon_t(1_{(1)})) \otimes 1_{(2)} = 1_{(1)}\overline{\varepsilon}_s(z) \otimes 1_{(2)} = 1_{(1)} \otimes 1_{(2)}z,$$

hence

$$\pi(mz \otimes n) = \pi(\overline{\varepsilon}_s(z)m \otimes n) = 1_{(1)}\overline{\varepsilon}_s(z)m \otimes 1_{(2)}n = 1_{(1)}m \otimes 1_{(2)}zn = \pi(m \otimes zn).$$

So π induces a map $\overline{\pi}: \ M \otimes_{H_t} N \to M \otimes_t N$, which is a left H_t-module isomorphism with inverse given by

$$\overline{\pi}^{-1}(1_{(1)}m \otimes 1_{(2)}n) = 1_{(1)}m \otimes_{H_t} 1_{(2)}n = m \otimes_{H_t} n.$$

Proposition 1.4. *Let H be a weak bialgebra. Then we have a monoidal category $({}_H\mathcal{M}, \otimes_t, H_t, a, l, r)$. The associativity constraints are the natural ones. The left and right unit constraints $l_M: \ H_t \otimes_t M \to M$ and $r_M: \ M \otimes_t H_t \to M$ and their inverses are given by the formulas*

$$l_M(1_{(1)} \rightharpoonup z \otimes 1_{(2)}m) = zm \ ; \ l_M^{-1}(m) = \varepsilon_t(1_{(1)}) \otimes 1_{(2)}m;$$

$$r_M(1_{(1)}m \otimes 1_{(2)} \rightharpoonup z) = \overline{\varepsilon}_s(z)m \ ; \ r_M^{-1}(m) = 1_{(1)}m \otimes 1_{(2)}.$$

Proof. This is a direct consequence of the observations made above. Let us check that

$$\begin{aligned}
l_M^{-1}(l_M(1_{(1)} \rightharpoonup z \otimes 1_{(2)}m)) &= l_M^{-1}(zm) = \varepsilon_t(1_{(1)}) \otimes 1_{(2)}zm \\
&= z\varepsilon_t(1_{(1)}) \otimes 1_{(2)}m \overset{(10)}{=} \varepsilon_t(z1_{(1)}) \otimes 1_{(2)}m \\
&= \varepsilon_t(1_{(1)}z) \otimes 1_{(2)}m = 1_{(1)} \rightharpoonup z \otimes 1_{(2)}m \\
l_M(l_M^{-1}(m)) &= l_M(\varepsilon_t(1_{(1)}) \otimes 1_{(2)}m) = m \\
r_M^{-1}(r_M(1_{(1)}m \otimes 1_{(2)} \rightharpoonup z)) &= r_M^{-1}(\overline{\varepsilon}_s(z)m) = 1_{(1)}\overline{\varepsilon}_s(z)m \otimes 1_{(2)} \\
&= 1_{(1)}m \otimes 1_{(2)}z = 1_{(1)}m \otimes 1_{(2)} \rightharpoonup z \\
r_M(r_M^{-1}(m)) &= r_M(1_{(1)}m \otimes 1_{(2)}) = \overline{\varepsilon}_s(1)m = m.
\end{aligned}$$

□

1.2. **Weak Hopf algebras.** A weak Hopf algebra is a weak bialgebra together with a map $S: H \to H$, called the antipode, satisfying

$$S * H = \varepsilon_s,\ H * S = \varepsilon_t,\ \text{and}\ S * H * S = S, \tag{13}$$

where $*$ is the convolution product. It follows immediately that

$$S = \varepsilon_s * S = S * \varepsilon_t. \tag{14}$$

If the antipode exists, then it is unique. We will always assume that S is bijective; if H is a finite dimensional weak Hopf algebra over a field, then S is automatically bijective (see [2, Theorem 2.10]).

Lemma 1.5. *Let H be a weak Hopf algebra. Then S is an anti-algebra and an anti-coalgebra morphism. For all $h, g \in H$, we have*

$$\varepsilon_t(hg) = \varepsilon_t(h\varepsilon_t(g)) = h_{(1)}\varepsilon_t(g)S(h_{(2)}); \tag{15}$$

$$\varepsilon_s(hg) = \varepsilon_s(\varepsilon_s(h)g) = S(g_{(1)})\varepsilon_s(h)g_{(2)}; \tag{16}$$

$$\Delta(\varepsilon_t(h)) = h_{(1)}S(h_{(3)}) \otimes \varepsilon_t(h_{(2)}) \tag{17}$$

$$\Delta(\varepsilon_s(h)) = \varepsilon_s(h_{(2)}) \otimes S(h_{(1)})h_{(3)}. \tag{18}$$

Lemma 1.6. *Let H be a weak Hopf algebra. For all $h \in H$, we have*

$$\varepsilon_t(h) = \varepsilon(S(h)1_{(1)})1_{(2)} = \varepsilon(1_{(2)}h)S(1_{(1)}) = S(\overline{\varepsilon}_s(h)) \tag{19}$$

$$\varepsilon_s(h) = 1_{(1)}\varepsilon(1_{(2)}S(h)) = \varepsilon(h1_{(1)})S(1_{(2)}) = S(\overline{\varepsilon}_t(h)). \tag{20}$$

Corollary 1.7. *Let H be a weak Hopf algebra. For all $h \in H$, we have*

$$\varepsilon_t(h_{(1)}) \otimes h_{(2)} = S(1_{(1)}) \otimes 1_{(2)}h\ ;\ h_{(1)} \otimes \varepsilon_s(h_{(2)}) = h1_{(1)} \otimes S(1_{(2)}). \tag{21}$$

Proposition 1.8. *Let H be a weak Hopf algebra. Then*

$$\varepsilon_t \circ S = \varepsilon_t \circ \varepsilon_s = S \circ \varepsilon_s\ ;\ \varepsilon_s \circ S = \varepsilon_s \circ \varepsilon_t = S \circ \varepsilon_t. \tag{22}$$

Corollary 1.9. *Let H be a weak Hopf algebra with bijective antipode. Then $S_{|H_t} = (\varepsilon_s)_{|H_t}$, and $S^{-1}_{|H_s} = (\overline{\varepsilon}_t)_{|H_s}$, so S restricts to an anti-algebra isomorphism $H_t \to H_s$.*

It follows that the separability idempotents of H_t and H_s are $e_t = S(1_{(1)}) \otimes 1_{(2)}$ and $e_s = 1_{(1)} \otimes S(1_{(2)})$. Consequently, we have the following formulas, for $z \in H_t$ and $y \in H_s$:

$$zS(1_{(1)}) \otimes 1_{(2)} = S(1_{(1)}) \otimes 1_{(2)}z; \tag{23}$$

$$y1_{(1)} \otimes 1_{(2)} = 1_{(1)} \otimes S^{-1}(y)1_{(2)}. \tag{24}$$

Applying $S^{-1} \otimes H$ to (23), we find

$$1_{(1)}S^{-1}(z) \otimes 1_{(2)} = 1_{(1)} \otimes 1_{(2)}z. \tag{25}$$

1.3. **The center of a monoidal category.** Let $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$ be a monoidal category. The weak left center $\mathcal{W}_l(\mathcal{C})$ is the category with the following objects and morphisms. An object is a couple $(M, \sigma_{M,-})$, with $M \in \mathcal{C}$ and $\sigma_{M,-}: M \otimes - \to - \otimes M$ a natural transformation, satisfying the following condition, for all $X, Y \in \mathcal{C}$:

$$(X \otimes \sigma_{M,Y}) \circ a_{X,M,Y} \circ (\sigma_{M,X} \otimes Y) = a_{X,Y,M} \circ \sigma_{M,X\otimes Y} \circ a_{M,X,Y}, \tag{26}$$

and such that $\sigma_{M,I}$ is the composition of the natural isomorphisms $M \otimes I \cong M \cong I \otimes M$. A morphism between $(M, \sigma_{M,-})$ and $(M', \sigma_{M',-})$ consists of $\vartheta: M \to M'$ in $\mathcal{C}$ such that

$$(X \otimes \vartheta) \circ \sigma_{M,X} = \sigma_{M',X} \circ (\vartheta \otimes X).$$

The left center $\mathcal{Z}_l(\mathcal{C})$ is the full subcategory of $\mathcal{W}_l(\mathcal{C})$ consisting of objects $(M,\sigma_{M,-})$ with $\sigma_{M,-}$ a natural isomorphism. $\mathcal{Z}_l(\mathcal{C})$ is a braided monoidal category. The tensor product is

$$(M,\sigma_{M,-})\otimes(M',\sigma_{M',-})=(M\otimes M',\sigma_{M\otimes M',-})$$

with

$$\sigma_{M\otimes M',X}=a_{X,M,M'}\circ(\sigma_{M,X}\otimes M')\circ a^{-1}_{M,X,M'}\circ(M\otimes\sigma_{M',X})\circ a_{M,M',X}, \tag{27}$$

and the unit is $(I,\sigma_{I,-})$, with

$$\sigma_{I,M}=r_M^{-1}\circ l_M. \tag{28}$$

The braiding c on $\mathcal{Z}_l(\mathcal{C})$ is given by

$$c_{M,M'}=\sigma_{M,M'}:\ (M,\sigma_{M,-})\otimes(M',\sigma_{M',-})\to(M',\sigma_{M',-})\otimes(M,\sigma_{M,-}). \tag{29}$$

$\mathcal{Z}_l(\mathcal{C})^{\rm in}$ will be our notation for the monoidal category $\mathcal{Z}_l(\mathcal{C})$, together with the inverse braiding $\tilde{c}$ given by $\tilde{c}_{M,M'}=c^{-1}_{M',M}=\sigma^{-1}_{M',M}$.
The right center $\mathcal{Z}_r(\mathcal{C})$ is defined in a similar way. An object is a couple $(M,\tau_{-,M})$, where $M\in\mathcal{C}$ and $\tau_{-,M}:\ -\otimes M\to M\otimes -$ is a family of natural isomorphisms such that $\tau_{-,I}$ is the natural isomorphism and

$$a^{-1}_{M,X,Y}\circ\tau_{X\otimes Y,M}\circ a^{-1}_{X,Y,M}=(\tau_{X,M}\otimes Y)\circ a^{-1}_{X,M,Y}\circ(X\otimes\tau_{Y,M}), \tag{30}$$

for all $X,Y\in\mathcal{C}$. A morphism between $(M,\tau_{-,M})$ and $(M',\tau_{-,M'})$ consists of $\vartheta:\ M\to M'$ in $\mathcal{C}$ such that

$$(\vartheta\otimes X)\circ\tau_{X,M}=\tau_{X,M'}\circ(X\otimes\vartheta),$$

for all $X\in\mathcal{C}$. $\mathcal{Z}_r(\mathcal{C})$ is a braided monoidal category. The unit is $(I,l_-^{-1}\circ r_-)$ and the tensor product is

$$(M,\tau_{-,M})\otimes(M',\tau_{-,M'})=(M\otimes M',\tau_{-,M\otimes M'})$$

with

$$\tau_{X,M\otimes M'}=a^{-1}_{M,M',X}\circ(M\otimes\tau_{X,M'})\circ a_{M,X,M'}\circ(\tau_{X,M}\otimes M')\circ a^{-1}_{X,M,M'}. \tag{31}$$

The braiding d is given by

$$d_{M,M'}=\tau_{M,M'}:\ (M,\tau_{-,M})\otimes(M',\tau_{-,M'})\to(M',\tau_{-,M'})\otimes(M,\tau_{-,M}). \tag{32}$$

$\mathcal{Z}_r(\mathcal{C})^{\rm in}$ is the monoidal category $\mathcal{Z}_r(\mathcal{C})$ with the inverse braiding $\tilde{d}$ given by $\tilde{d}_{M,M'}=d^{-1}_{M',M}=\tau^{-1}_{M',M}$.
For details in the case where $\mathcal{C}$ is a strict monoidal category, we refer to [12, Theorem XIII.4.2]. The results remain valid in the case of an arbitrary monoidal category, since every monoidal category is equivalent to a strict one. Recall the following result from [5].

Proposition 1.10. *Let $\mathcal{C}$ be a monoidal category. Then we have an isomorphism of braided monoidal categories $F:\ \mathcal{Z}_l(\mathcal{C})\to\mathcal{Z}_r(\mathcal{C})^{\rm in}$, given by*

$$F(M,\sigma_{M,-})=(M,\sigma^{-1}_{M,-})\ \text{and}\ F(\vartheta)=\vartheta.$$

We have a second monoidal structure on $\mathcal{C}$, defined as follows:

$$\overline{\mathcal{C}}=(\mathcal{C},\overline{\otimes}=\otimes\circ\tau,I,\overline{a},r,l)$$

with $\tau:\ \mathcal{C}\times\mathcal{C}\to\mathcal{C}\times\mathcal{C}$, $\tau(M,N)=(N,M)$ and $\overline{a}$ defined by $\overline{a}_{M,N,X}=a^{-1}_{X,N,M}$.
If c is a braiding on $\mathcal{C}$, then $\overline{c}$, given by $\overline{c}_{M,N}=c_{N,M}$ is a braiding on $\overline{\mathcal{C}}$. In [5], the following obvious result was stated.

Proposition 1.11. *Let $\mathcal{C}$ be a monoidal category. Then*

$$\overline{\mathcal{Z}_l(\mathcal{C})} \cong \mathcal{Z}_r(\overline{\mathcal{C}}) \ ; \ \overline{\mathcal{Z}_r(\mathcal{C})} \cong \mathcal{Z}_l(\overline{\mathcal{C}})$$

as braided monoidal categories.

1.4. **Weak entwining structures and weak smash products.** The results in this Section are taken from [6]. Let A be a ring without unit. $e \in A$ is called a preunit if $ea = ae = ae^2$, for all $a \in A$. Then map $p: A \to A$, $p(a) = ae$, satisfies the following properties: $p \circ p = p$ and $p(ab) = p(a)p(b)$. Then $\overline{A} = \mathrm{Coim}(p)$ is a ring with unit $\overline{e}$ and $\underline{A} = \mathrm{Im}\,(p)$ is a ring with unit e^2. p induces a ring isomorphism $\overline{A} \to \underline{A}$.
Let k be a commutative ring, A, B k-algebras with unit, and $R: B \otimes A \to A \otimes B$ a k-linear map. We use the notation

$$R(b \otimes a) = a_R \otimes b_R = a_r \otimes b_r, \tag{33}$$

where the summation is implicitely understood. $A\#_R B$ is the k-algebra $A \otimes B$ with newly defined multiplication

$$(a\#b)(c\#d) = ac_R\#b_R d.$$

(A, B, R) is called a weak smash product structure if $A\#_R B$ is an associative k-algebra with preunit $1_A\#1_B$. The multiplication is associative if and only if

$$R(bd \otimes a) = a_{Rr} \otimes b_r d_R \text{ and } R(b \otimes ac) = a_R c_r \otimes b_{Rr},$$

for all $a, c \in A$ and $b, d \in B$. $1_A\#1_B$ is a preunit if and only if

$$R(1_B \otimes a) = a(1_A)_R \otimes (1_B)_R \text{ and } R(b \otimes 1_A) = (1_A)_R \otimes (1_B)_R b.$$

A left-right weak entwining structure is a triple (A, C, ψ), where A is an algebra, C is a coalgebra, and $\psi: A \otimes C \to A \otimes C$ is a k-linear map satisfying the conditions

$$a_\psi \otimes \Delta(c^\psi) = a_{\psi\Psi} \otimes c_{(1)}^{\Psi} \otimes c_{(2)}^{\psi} \ ; \ (ab)_\psi \otimes c^\psi = a_\psi b_\Psi \otimes c^{\Psi\psi};$$

$$1_\psi \otimes c^\psi = \varepsilon(c_{(1)}^{\psi})1_\psi \otimes c_{(2)} \ ; \ a_\psi \varepsilon(c^\psi) = \varepsilon(c^\psi)a1_\psi.$$

Here we use the notation (with summation implicitely understood):

$$\psi(a \otimes c) = a_\psi \otimes c^\psi.$$

An entwined module is a k-module M with a left A-action and a right C-coaction such that

$$\rho(am) = a_\psi m_{[0]} \otimes m_{[1]}^{\psi}.$$

The category of entwined modules and left A-linear right C-colinear maps is denoted by ${}_A\mathcal{M}(\psi)^C$.
Let H be a weak bialgebra, and A a right H-comodule, which is also an algebra with unit. A is called a right H-comodule algebra if $\rho(a)\rho(b) = \rho(ab)$ and $1_{[0]} \otimes \varepsilon_t(1_{[1]}) = \rho(1)$.

From [1], we recall the following definitions. Let C be a left H-module which is also a coalgebra with counit. C is called a left H-comodule algebra if $\Delta_C(hc) = \Delta_H(h)\Delta_C(c)$ and

$$\varepsilon_C(hkc) = \varepsilon_H(hk_{(2)})\varepsilon_C(k_{(1)}c), \tag{34}$$

for all $c \in C$ and $h, k \in H$. Several equivalent definitions are given in [6, Sec. 4]. We then call (H, A, C) a left-right weak Doi-Hopf datum. A weak Doi-Hopf

module over (H,A,C) is a k-module M with a left A-action and a right C-coaction, satisfying the following compatibility relation, for all $m \in M$ and $a \in A$:

$$\rho(am) = a_{[0]}m_{[0]} \otimes a_{[1]}m_{[1]}. \tag{35}$$

The category of weak Doi-Hopf modules over (H,A,C) and left A-linear right C-colinear maps is denoted by ${}_A\mathcal{M}(H)^C$.
Let (H,A,C) be a weak left-right Doi-Hopf datum, and consider the map

$$\psi:\ A\otimes C \to A\otimes C,\ \psi(a\otimes c) = a_{[0]}\otimes a_{[1]}c.$$

Then (A,C,ψ) is a weak left-right entwining structure, and we have an isomorphism of categories ${}_A\mathcal{M}(H)^C \cong {}_A\mathcal{M}(\psi)^C$.

Let (A,C,ψ) be a weak left-right entwining structure, and assume that C is finitely generated projective as a k-module, with finite dual basis $\{(c_i,c_i^*)\mid i=1,\cdots,n\}$. Then we have a weak smash product structure (A,C^*,R), with $R:\ C^*\otimes A\to A\otimes C^*$ given by

$$R(c^*\otimes a) = \textstyle\sum_i \langle c^*, c_i^\psi\rangle a_\psi \otimes c_i^*. \tag{36}$$

We have an isomorphism of categories

$$F:\ {}_A\mathcal{M}(\psi)^C \to {}_{\overline{A\#_R C^*}}\mathcal{M}, \tag{37}$$

defined also follows: $F(M)=M$ as a k-module, with action $[a\#c^*]\cdot m = \langle c^*, m_{[1]}\rangle am_{[0]}$. Details can be found in [6, Theorem 3.4].

2. Yetter-Drinfeld modules over weak Hopf algebras

Let H be a weak bialgebra. A left-left Yetter-Drinfeld module is a k-module with a left H-action and a left H-coaction such that the following conditions hold, for all $m\in M$ and $h\in H$:

$$\lambda(m) = m_{[-1]}\otimes m_{[0]} \in H\otimes_t M; \tag{38}$$

$$h_{(1)}m_{[-1]}\otimes h_{(2)}m_{[0]} = (h_{(1)}m)_{[-1]}h_{(2)}\otimes (h_{(1)}m)_{[0]}. \tag{39}$$

We will now state some equivalent definitions. First we will rewrite the counit property for Yetter-Drinfeld modules.

Lemma 2.1. *Let H be a weak bialgebra, and $\lambda:\ M\to H\otimes_t M$, $\rho(m) = m_{[-1]}\otimes m_{[0]}$ a k-linear map. Then*

$$\varepsilon(m_{[-1]})m_{[0]} = \varepsilon_t(m_{[-1]})m_{[0]}. \tag{40}$$

Consequently, in the definition of a Yetter-Drinfeld module, the counit property $\varepsilon(m_{[-1]})m_{[0]} = m$ can be replaced by $\varepsilon_t(m_{[-1]})m_{[0]} = m$.

Proof.

$$\varepsilon_t(m_{[-1]})m_{[0]} = \varepsilon(1_{(1)}m_{[-1]})1_{(2)}m_{[0]} = \varepsilon(m_{[-1]})m_{[0]}.$$

□

In the case of a weak Hopf algebra, the compatibility relation (39) can also be restated:

Proposition 2.2. *(cf.* [16, Remark 2.6]*) Let H be a weak Hopf algebra, and M a k-module, with a left H-action and a left H-coaction. M is a Yetter-Drinfeld module if and only if*

$$\lambda(hm) = h_{(1)}m_{[-1]}S(h_{(3)}) \otimes h_{(2)}m_{[0]}. \tag{41}$$

Proof. Let M be a Yetter-Drinfeld module. Then we compute

$$\begin{aligned} h_{(1)}m_{[-1]}S(h_{(3)}) \otimes h_{(2)}m_{[0]} &= (h_{(1)}m)_{[-1]}h_{(2)}S(h_{(3)}) \otimes (h_{(1)}m)_{[0]} \\ &= (h_{(1)}m)_{[-1]}\varepsilon_t(h_{(2)}) \otimes (h_{(1)}m)_{[0]} \overset{(3)}{=} (1_{(1)}hm)_{[-1]}1_{(2)} \otimes (1_{(1)}hm)_{[0]} \\ &\overset{(39)}{=} 1_{(1)}(hm)_{[-1]} \otimes 1_{(2)}(hm)_{[0]} \overset{(38)}{=} (hm)_{[-1]} \otimes (hm)_{[0]} = \lambda(hm). \end{aligned}$$

Conversely, assume that (41) holds for all $h \in H$ and $m \in M$. Taking $h = 1$ in (41), we find

$$\begin{aligned} \lambda(m) &= 1_{(1)}m_{[-1]}S(1_{(3)}) \otimes 1_{(2)}m_{[0]} \\ &= 1_{(1)}m_{[-1]}S(1_{(2')}) \otimes 1_{(2)}1_{(1')}m_{[0]} \in H \otimes_t M \end{aligned}$$

and

$$\begin{aligned} \lambda(m) &= 1_{(1)}m_{[-1]}S(1_{(3)}) \otimes 1_{(2)}m_{[0]} = 1_{(1)}m_{[-1]}S(1_{(2')}) \otimes 1_{(1')}1_{(2)}m_{[0]} \\ &= m_{[-1]}S(1_{(2')}) \otimes 1_{(1')}m_{[0]}. \end{aligned} \tag{42}$$

Now

$$\begin{aligned} (h_{(1)}m)_{[-1]}h_{(2)} \otimes (h_{(1)}m)_{[0]} &\overset{(41)}{=} h_{(1)}m_{[-1]}S(h_{(3)})h_{(4)} \otimes h_{(2)}m_{[0]} \\ &= h_{(1)}m_{[-1]}\varepsilon_s(h_{(3)}) \otimes h_{(2)}m_{[0]} \overset{(21)}{=} h_{(1)}m_{[-1]}S(1_{(2)}) \otimes h_{(2)}1_{(1)}m_{[0]} \\ &\overset{(42)}{=} h_{(1)}m_{[-1]} \otimes h_{(2)}m_{[0]}, \end{aligned}$$

as needed. □

Corollary 2.3. *Let M be a left-left Yetter-Drinfeld module. For all $y \in H_s$, $z \in H_t$ and $m \in M$, we have*

$$\lambda(zm) = zm_{[-1]} \otimes m_{[0]} \ ; \ \lambda(ym) = m_{[-1]}S(y) \otimes m_{[0]}. \tag{43}$$

Proof.

$$\begin{aligned} \lambda(zm) &\overset{(6,41)}{=} 1_{(1)}zm_{[-1]}S(1_{(3)}) \otimes 1_{(2)}m_{[0]} \\ &\overset{(8)}{=} z1_{(1)}m_{[-1]}S(1_{(3)}) \otimes 1_{(2)}m_{[0]} \overset{(41)}{=} zm_{[-1]} \otimes m_{[0]}. \end{aligned}$$

The other assertion is proved in a similar way. □

Corollary 2.4. *Let M be a left-left Yetter-Drinfeld module over a weak Hopf algebra with bijective antipode. Then we have the following identities, for all $m \in M$:*

$$1_{(1)}m_{[0]} \otimes 1_{(2)}S^{-1}(m_{[-1]}) = m_{[0]} \otimes S^{-1}(m_{[-1]}); \tag{44}$$

$$\varepsilon_s(S^{-2}(m_{[-1]}))m_{[0]} = m. \tag{45}$$

Proof. Apply S^{-1} to the first factor of (42), and then switch the two tensor factors. Then we obtain (44). (45) is proved as follows:

$$\begin{aligned} m = \varepsilon_t(m_{[-1]})m_{[0]} &\overset{(40)}{=} \varepsilon(m_{[-1]})m_{[0]} = \varepsilon(S^{-1}(m_{[-1]}))m_{[0]} \\ &\overset{(44)}{=} \varepsilon(1_{(2)}S^{-1}(m_{[-1]}))1_{(1)}m_{[0]} \overset{(20)}{=} \varepsilon_s(S^{-2}(m_{[-1]}))m_{[0]}. \end{aligned}$$

□

The category of left-left Yetter-Drinfeld modules and left H-linear, left H-colinear maps will be denoted by ${}^H_H\mathcal{YD}$.

Example 2.5. Let G be a groupoid, and kG the corresponding groupoid algebra. Then kG is a weak Hopf algebra. Let M be a left-left Yetter-Drinfeld module. Then M is a kG-comodule, so M is graded by the set G, that is

$$M = \bigoplus_{\sigma \in G_1} M_\sigma,$$

and $\lambda(m) = \sigma \otimes m$ if and only if $m \in M_\sigma$, or $\deg(m) = \sigma$.
Recall that the unit element of kG is $1 = \sum_{x \in G_0} x$, where x is the identity morphism of the object $x \in G_0$. Take $m \in M_\sigma$. Using (41), we find

$$\lambda(m) = \lambda(1m) = \sum_{x \in G} x\sigma x \otimes xm = 0,$$

unless $s(\sigma) = \tau(\sigma) = x$. So we have

$$M = \bigoplus_{\substack{\sigma \in G_1 \\ s(\sigma)=t(\sigma)}} M_\sigma.$$

Take $m \in M_\sigma$, with $s(\sigma) = \tau(\sigma)$, and $\tau \in G_1$. It follows from (41) that $\lambda(\tau m) = \tau\sigma\tau^{-1} \otimes \tau m = 0$, unless $s(\tau) = x$. If $s(\tau) = x$, then $\deg(\tau m) = \tau\sigma\tau^{-1}$.

Theorem 2.6. *Let H be a weak bialgebra. Then the category ${}^H_H\mathcal{YD}$ is isomorphic to the weak left center $\mathcal{W}_l({}_H\mathcal{M})$ of the category of left H-modules. If H is a weak Hopf algebra with bijective antipode, then ${}^H_H\mathcal{YD}$ is isomorphic to the left center $\mathcal{Z}_l({}_H\mathcal{M})$*

Proof. We will restrict to a brief description of the connecting functors; for more detail (in the left-right case), we refer to [16, Lemma 4.3]. Take $(M, \sigma_{M,-}) \in \mathcal{W}_l({}_H\mathcal{M})$. For each left H-module V, we have a map $\sigma_{M,V}: M \otimes_t V \to V \otimes_t M$ in ${}_H\mathcal{M}$. We will show that the map

$$\lambda: M \to H \otimes_t M, \ \lambda(m) = \sigma_{M,H}(1_{(1)}m \otimes 1_{(2)}) = m_{[-1]} \otimes m_{[0]}$$

makes M into a Yetter-Drinfeld module. Conversely, let (M, λ) is a Yetter-Drinfeld module; a natural transformation σ is then defined by the formula

$$\sigma_{M,V}(1_{(1)}m \otimes 1_{(2)}v) = m_{[-1]}v \otimes m_{[0]}. \tag{46}$$

Straightforward computations show that $(M, \sigma) \in \mathcal{W}_l({}_H\mathcal{M})$. If H is a Hopf algebra with invertible antipode, then the inverse of $\sigma_{M,V}$ is

$$\sigma^{-1}_{M,V}(1_{(1)}v \otimes 1_{(2)}m) = m_{[0]} \otimes S^{-1}(m_{[-1]})v. \tag{47}$$

□

From now on, we assume that H is a weak Hopf algebra with bijective antipode. Since the left center of a monoidal category is a braided monoidal category, it follows from Theorem 2.6 that ${}^H_H\mathcal{YD}$ is a braided monoidal category; a direct but long proof can be given: see [16, Prop. 2.7]. The monoidal structure can be computed using (27). Take $M, N \in {}^H_H\mathcal{YD}$, the H-coaction on $M \otimes_t N$ is given by the formula

$$\lambda(1_{(1)}m \otimes 1_{(2)}n) = ((\sigma_{M,H} \otimes N) \circ (M \otimes \sigma_{N,H}))(1_{(1')}(1_{(1)}m \otimes 1_{(2)}n) \otimes 1_{(2')}).$$

Observe that

$$\begin{aligned} x &= 1_{(1')}(1_{(1)}m\otimes 1_{(2)}n)\otimes 1_{(2')}\\ &= 1_{(1')}1_{(1)}m\otimes 1_{(1'')}1_{(2')}1_{(2)}n\otimes 1_{(2'')} = 1_{(1)}m\otimes 1_{(1'')}1_{(2)}n\otimes 1_{(2'')}, \end{aligned}$$

so that

$$\begin{aligned} (M\otimes\sigma_{N,H})(x) &= 1_{(1)}m\otimes (1_{(2)}n)_{[-1]}\otimes (1_{(2)}n)_{[0]}\\ &= 1_{(1)}m\otimes 1_{(2)}n_{[-1]}S(1_{(4)})\otimes 1_{(3)}n_{[0]}\\ &= 1_{(1)}m\otimes 1_{(2)}1_{(1')}n_{[-1]}S(1_{(3')})\otimes 1_{(2')}n_{[0]}\\ &= 1_{(1)}m\otimes 1_{(2)}n_{[-1]}\otimes n_{[0]} \end{aligned}$$

and

$$\lambda(1_{(1)}m\otimes 1_{(2)}n) = m_{[-1]}n_{[-1]}\otimes m_{[0]}\otimes n_{[0]}. \tag{48}$$

We compute the left H-coaction on H_t using (28) and (46). For any $z\in H_t$, this gives

$$\begin{aligned} \lambda(z) &= \sigma_{H_t,H}((1_{(1)}\rightharpoonup z)\otimes 1_{(2)}) = r_M^{-1}(l_M((1_{(1)}\rightharpoonup z)\otimes 1_{(2)}))\\ &= r_M^{-1}(z) = 1_{(1)}z\otimes 1_{(2)} = \Delta(z). \end{aligned} \tag{49}$$

The braiding and its inverse are given by the formulas

$$\sigma_{M,N}(1_{(1)}m\otimes 1_{(2)}n) = m_{[-1]}n\otimes m_{[0]}\ ;\ \sigma^{-1}_{M,N}(1_{(1)}n\otimes 1_{(2)}m) = m_{[0]}\otimes S^{-1}(m_{[-1]})n.$$

A left-right Yetter-Drinfeld module is a k-module with a left H-action and a right H-coaction such that the following conditions hold, for all $m\in M$ and $h\in H$:

$$\rho(m) = m_{[0]}\otimes m_{[1]}\in M\otimes_t H; \tag{50}$$

$$h_{(1)}m_{[0]}\otimes h_{(2)}m_{[1]} = (h_{(2)}m)_{[0]}\otimes (h_{(2)}m)_{[1]}h_{(1)}. \tag{51}$$

The category of left-right Yetter-Drinfeld modules and left H-linear right H-colinear maps is denoted by ${}_H\mathcal{YD}^H$.

Proposition 2.7. *Let H be a weak Hopf algebra with bijective antipode. Then the category ${}_H\mathcal{YD}^H$ is isomorphic to the right center $\mathcal{Z}_r({}_H\mathcal{M})$.*

Proof. Take $(M,\tau_{-,M})\in\mathcal{Z}_r({}_H\mathcal{M})$. We know from Proposition 1.10 that $(M,\sigma_{M,-}=\tau^{-1}_{-,M})\in\mathcal{Z}_l({}_H\mathcal{M})$. Take the corresponding left-left Yetter-Drinfeld (M,λ), as in Theorem 2.6, and define $\rho:\ M\to M\otimes H$ by

$$\rho(m) = m_{[0]}\otimes m_{[1]} = m_{[0]}\otimes S^{-1}(m_{[-1]}). \tag{52}$$

It follows from (44) that $\rho(m)\in M\otimes_t H$. The coassociativity of ρ follows immediately from the coassociativity of λ and the anti-comultiplicativity of S^{-1}. Also

$$\varepsilon(m_{[1]})m_{[0]} = \varepsilon(S^{-1}(m_{[-1]}))m_{[0]} = \varepsilon(m_{[-1]})m_{[0]} = m.$$

From (47), it follows that

$$\tau_{V,M}(1_{(1)}v\otimes 1_{(2)}m) = m_{[0]}\otimes m_{[1]}v. \tag{53}$$

In particular, $\tau_{M,H}(1_{(1)}\otimes 1_{(2)}m) = \rho(m)$, and the fact that $\tau_{M,H}$ is left H-linear implies (51). Hence (M,ρ) is a left-right Yetter-Drinfeld module.
Conversely, if (M,ρ) is a left-right Yetter-Drinfeld module, then $(M,\tau_{-,M})$, with τ defined by (53) is an object of $\mathcal{Z}_r({}_H\mathcal{M})$. $\square$

Corollary 2.8. *Let M be a k-module with a left H-action and a right H-coaction. Then M is a left-right Yetter-Drinfeld module if and only if*

$$\rho(hm) = h_{(2)}m_{[0]} \otimes h_{(3)}m_{[1]}S^{-1}(h_{(1)}). \tag{54}$$

Corollary 2.9. *Let M be a left-right Yetter-Drinfeld module. For all $y \in H_s$, $z \in H_t$ and $m \in M$, we have that*

$$\rho(ym) = m_{[0]} \otimes ym_{[1]} \ ; \ \rho(zm) = m_{[0]} \otimes m_{[1]}S^{-1}(z). \tag{55}$$

Corollary 2.10. *Let M be a left-right Yetter-Drinfeld module. Then*

$$1_{(2)}m_{[0]} \otimes m_{[1]}S^{-1}(1_{(1)}) = \rho(m), \tag{56}$$

for all $m \in M$.

Proof. Apply $S^{-1} \otimes M$ to $\lambda(m) = 1_{(1)}S(m_{[1]}) \otimes 1_{(2)}m_{[0]}$. □

Corollary 2.11. *The category ${}_H\mathcal{YD}^H$ is a braided monoidal category, isomorphic to ${}^H_H\mathcal{YD}^{\rm in}$.*

In a similar way, we can introduce right-right and right-left Yetter-Drinfeld modules. The categories $\mathcal{YD}_H^H$ and ${}^H\mathcal{YD}_H$ of right-right and right-left Yetter-Drinfeld modules are isomorphic to the right and left center of $\mathcal{M}_H$. Let us summarize the results.

A right-right Yetter-Drinfeld module is a k-module M with a right H-action and a right H-coaction such that

$$\rho(m) = m_{[0]} \otimes m_{[1]} \in M \otimes_s H; \tag{57}$$

$$m_{[0]}h_{(1)} \otimes m_{[1]}h_{(2)} = (mh_{(2)})_{[0]} \otimes h_{(1)}(mh_{(2)})_{[1]}; \tag{58}$$

or, equivalently,

$$\rho(mh) = m_{[0]}h_{(2)} \otimes S(h_{(1)})m_{[1]}h_{(3)}. \tag{59}$$

The counit condition $m = \varepsilon(m_{[1]})m_{[0]}$ is equivalent to

$$m = m_{[0]}\varepsilon(m_{[1]}).$$

The natural isomorphism $\tau_{-,M}$ corresponding to $(M,\rho) \in \mathcal{YD}_H^H$ and its inverse are given by the formulas

$$\tau_{M,V}(v1_{(1)} \otimes m1_{(2)}) = m_{[0]} \otimes mv_{[1]} \ ; \ \tau^{-1}_{M,V}(m1_{(1)} \otimes v1_{(2)}) = vS^{-1}(m_{[1]}) \otimes m_{[0]}. \tag{60}$$

Furthermore

$$m_{[0]}\varepsilon_t(S^{-2}(m_{[1]})) = m,$$

and $S^{-1}(m_{[1]}) \otimes m_{[0]} \in H \otimes_s M$.

The monoidal structure on $\mathcal{YD}_H^H$ is given by the formula

$$\rho(m1_{(1)} \otimes n1_{(2)}) = m_{[0]} \otimes n_{[0]} \otimes m_{[1]}n_{[1]}.$$

The braiding is given by (60). The category $\mathcal{YD}_H^H$ is isomorphic as a braided monoidal category to $\mathcal{Z}_r(\mathcal{M}_H)$.

Let M be a right H-module and a left H-comodule. M is a right-left Yetter-Drinfeld module if one of the three following equivalent conditions is satisfied, for all $m \in M$ and $h \in H$:

1) $\lambda(m) \in H \otimes_s M$ and

$$h_{(2)}(mh_{(1)})_{[0]} \otimes (mh_{(1)})_{[1]} = m_{[-1]}h_{(1)} \otimes m_{[0]}h_{(2)},$$

2) $\lambda(mh) = S^{-1}(h_{(3)})m_{[-1]}h_{(1)} \otimes m_{[0]}h_{(2)}$;
3) (M,ρ), with $\rho(m) = m_{[0]} \otimes S(m_{[-1]})$ is a right-right Yetter-Drinfeld module.

The category of right-left Yetter-Drinfeld modules, ${}^H\mathcal{YD}_H$, is a braided monoidal category. The monoidal structure and the braiding are given by

$$\lambda(m1_{(1)} \otimes n1_{(2)}) = m_{[-1]}n_{[-1]} \otimes m_{[0]} \otimes n_{[0]};$$

$$\sigma_{M,N}(m1_{(1)} \otimes n1_{(2)}) = nm_{[-1]} \otimes m_{[0]}.$$

As a braided monoidal category, ${}^H\mathcal{YD}_H$ is isomorphic to $\mathcal{Z}_l(\mathcal{M}_H)$ and $(\mathcal{YD}^H_H)^{\mathrm{in}}$.

The antipode $S:\ H \to H^{\mathrm{op,cop}}$ is an isomorphism of weak Hopf algebras. Observe that the target map of $H^{\mathrm{op,cop}}$ is ε_s, and that its source map is ε_t. Thus S induces an isomorphism between the monoidal categories ${}_H\mathcal{M}$ and ${}_{H^{\mathrm{op,cop}}}\mathcal{M}$. We also have a monoidal isomorphism $F:\ {}_{H^{\mathrm{op,cop}}}\mathcal{M} \to \overline{\mathcal{M}}_H$, given by

$$F(M) = M,\ mh = h^{\mathrm{op,cop}}m.$$

indeed, in ${}_{H^{\mathrm{op,cop}}}\mathcal{M}$, $M \otimes_t N$ is generated by elements of the form $1_{(2)}m \otimes 1_{(1)}n$, and $F(M \otimes_t N)$ is generated by elements of the form $m1_{(2)} \otimes n1_{(1)}$. $F(N) \otimes_s F(M)$ is generated by elements of the form $n1_{(1)} \otimes m1_{(2)}$, and it follows that the switch map is an isomorphism $F(M \otimes_t N) \to F(N) \otimes_s F(M)$. We conclude from Proposition 1.11 that we have isomorphisms of braided monoidal categories

$${}^H_H\mathcal{YD} \cong \mathcal{Z}_l({}_H\mathcal{M}) \cong \mathcal{Z}_l({}_{H^{\mathrm{op,cop}}}\mathcal{M}) \cong \mathcal{Z}_l(\overline{\mathcal{M}}_H) \cong \overline{\mathcal{Z}_r(\mathcal{M}_H)} \cong \overline{\mathcal{YD}}^H_H.$$

This isomorphism can be described explicitely as follows:

$$F:\ {}^H_H\mathcal{YD} \to \overline{\mathcal{YD}}^H_H,\ F(M) = M,$$

with

$$m \cdot h = S^{-1}(h)m\ ;\ \rho(m) = m_{[0]} \otimes S(m_{[-1]}).$$

We summarize our results as follows:

Theorem 2.12. *Let H be a weak Hopf algebra with bijective antipode. Then we have the following isomorphisms of braided monoidal categories:*

$${}^H_H\mathcal{YD} \cong {}_H\mathcal{YD}^{H^{\mathrm{in}}} \cong \overline{\mathcal{YD}^H_H} \cong \overline{{}^H\mathcal{YD}_H}^{\mathrm{in}}.$$

3. Yetter-Drinfeld modules are Doi-Hopf modules

It was shown in [7] that Yetter-Drinfeld modules (over a classical Hopf algebra) can be considered as Doi-Hopf modules, and, a fortiori, as entwined modules, and as comodules over a coring (see [4]). Weak Doi-Hopf modules were introduced by Böhm [1], and they are special cases of weak entwined modules (see [6]), and these are in turn examples of comodules over a coring (see [4]). In this Section, we will show that Yetter-Drinfeld modules over weak Hopf algebras are special cases of weak Doi-Hopf modules. We will discuss the left-right case.

Proposition 3.1. *Let H be a weak Hopf algebra with a bijective antipode. Then H is a right $H \otimes H^{\mathrm{op}}$-comodule algebra, with H-coaction*

$$\rho(h) = h_{(2)} \otimes S^{-1}(h_{(1)}) \otimes h_{(3)}.$$

Proof. It is easy to verify that H is a right $H \otimes H^{\mathrm{op}}$-comodule and that $\rho(hk) = \rho(h)\rho(k)$. Recall that $H_t = \mathrm{Im}\,(\varepsilon_t) = \mathrm{Im}\,(\overline{\varepsilon}_t)$. The target map of $H^{\mathrm{op}} \otimes H$ is $\overline{\varepsilon}_t \otimes \varepsilon_t$. We now have

$$\begin{aligned} 1_{[0]} \otimes (\overline{\varepsilon}_t \otimes \varepsilon_t)(1_{[1]}) &= 1_{(2)}1_{(1')} \otimes \overline{\varepsilon}_t(S^{-1}(1_{(1)})) \otimes \varepsilon_t(1_{(2')}) \\ &= 1_{(2)}1_{(1')} \otimes S^{-1}(1_{(1)}) \otimes 1_{(2')} = \rho(1), \end{aligned}$$

where we used the fact that $S^{-1}(1_{(1)}) \otimes 1_{(2)} \in H_t \otimes H_t$. □

Proposition 3.2. *Let H be a weak Hopf algebra with a bijective antipode. Then H is a left $H^{\mathrm{op}} \otimes H$-module coalgebra with left action*

$$(k \otimes h) \triangleright c = hck.$$

Proof. We easily compute that

$$\begin{aligned} &\varepsilon((m \otimes l)(k_{(2)} \otimes h_{(2)}))\varepsilon((k_{(1)} \otimes h_{(1)}) \triangleright c) \\ &= \varepsilon(k_{(2)}m)\varepsilon(lh_{(2)})\varepsilon(h_{(1)}ck_{(1)}) \\ &= \varepsilon(lhckm) = \varepsilon(((m \otimes l)(k \otimes h)) \triangleright c). \end{aligned}$$

The other conditions are easily verified. □

Corollary 3.3. *Let H be a weak Hopf algebra with bijective antipode. Then we have a weak Doi-Hopf datum $(H^{\mathrm{op}} \otimes H, H, H)$ and the categories ${}_H\mathcal{M}(H^{\mathrm{op}} \otimes H)^H$ and ${}_H\mathcal{YD}^H$ are isomorphic.*

Proof. The compatibility relation (35) reduces to (54). □

As we have seen in Section 1.4, weak Doi-Hopf modules are special cases of entwined modules. The entwining map $\psi :\; H \otimes H \to H \otimes H$ corresponding to the weak Doi-Hopf datum $(H^{\mathrm{op}} \otimes H, H, H)$ is given by

$$\psi(h \otimes k) = h_{(2)} \otimes h_{(3)}kS^{-1}(h_{(1)}). \tag{61}$$

4. The Drinfeld double

Now we consider the particular case where H is finitely generated and projective as a k-module, with finite dual basis $\{(h_i, h_i^*) \mid i = 1, \cdots, n\}$. Then H^* is also a weak Hopf algebra, in view of the selfduality of the axioms of a weak Hopf algebra. Recall that the comultiplication is given by the formula $\langle \Delta(h^*), h \otimes k\rangle = \langle h^*, hk\rangle$; the counit is evaluation at 1. Also recall that H^* is an H-bimodule, with left and right H-action

$$\langle h \rightharpoonup h^* \leftharpoonup k, l\rangle = \langle h^*, klh\rangle,$$

or

$$h \rightharpoonup h^* \leftharpoonup k = \langle h^*_{(1)}, k\rangle\langle h^*_{(3)}, h\rangle h^*_{(2)}. \tag{62}$$

Using (36), we find a weak smash product structure (H, H^*, R), with $R :\; H^* \otimes H \to H \otimes H^*$ given by

$$\begin{aligned} R(h^* \otimes h) &= \textstyle\sum_i \langle h^*, h_{(3)}h_iS^{-1}(h_{(1)})\rangle h_{(2)} \otimes h_i^* \\ &= \textstyle\sum_i \langle S^{-1}(h_{(1)}) \rightharpoonup h^* \leftharpoonup h_{(3)}, h_i\rangle h_{(2)} \otimes h_i^* \\ &= h_{(2)} \otimes \Big(S^{-1}(h_{(1)}) \rightharpoonup h^* \leftharpoonup h_{(3)}\Big). \end{aligned} \tag{63}$$

From Section 1.4, we know that $H\#_R H^*$, which we will also denote by $H\bowtie H^*$, is an associative algebra with preunit $1\#\varepsilon$. Using (33), we compute the multiplication rule on $H\bowtie H^*$.

$$\begin{aligned}(h\bowtie h^*)(k\bowtie k^*) &= \textstyle\sum_i\langle h^*, k_{(3)}h_iS^{-1}(k_{(1)})\rangle hk_{(2)}\bowtie h_i^**k^* \\ &= hk_{(2)}\bowtie (S^{-1}(k_{(1)})\rightharpoonup h^*\leftharpoonup k_{(3)})*k^* \qquad (64)\\ &= hk_{(2)}\bowtie\langle h^*_{(1)},k_{(3)}\rangle\langle h^*_{(3)},S^{-1}(k_{(1)})\rangle h^*_{(2)}*k^*. \qquad (65)\end{aligned}$$

We have a projection $p:\ H\bowtie H^*\to H\bowtie H^*$,

$$p(h\bowtie h^*)=(1\bowtie\varepsilon)(h\bowtie h^*)=(h\bowtie h^*)(1\bowtie\varepsilon)=(h\bowtie h^*)(1\bowtie\varepsilon)^2,$$

and $D(H)=\overline{H\bowtie H^*}=(H\bowtie H^*)/\mathrm{Ker}\, p$ is a k-algebra with unit $[1\bowtie\varepsilon]$, which we call the *Drinfeld double* of H. $D(H)$ is also isomorphic to $\underline{H\bowtie H^*}=\mathrm{Im}\,(p)$, which is a k-algebra with unit $(1\bowtie\varepsilon)^2$. Observe that the multiplication rule (65) is the same as in [1, 16]. We show that the ideal J that is divided out in [1, 16] is equal to $\mathrm{Ker}\, p$, and this will imply that $D(H)$ is equal to the Drinfeld double introduced in [1, 16]. We first need some Lemmas.

Lemma 4.1. *Let H a weak bialgebra. For all $h^*\in H^*$, $y\in H_s$ and $z\in H_t$, we have*

$$h^**(y\rightharpoonup\varepsilon) = \langle h^*_{(2)},y\rangle h^*_{(1)}=y\rightharpoonup h^* \qquad (66)$$
$$h^**(\varepsilon\leftharpoonup y) = \langle h^*_{(1)},y\rangle h^*_{(2)}=h^*\leftharpoonup y \qquad (67)$$
$$(z\rightharpoonup\varepsilon)*h^* = \langle h^*_{(2)},z\rangle h^*_{(1)}=z\rightharpoonup h^* \qquad (68)$$
$$(\varepsilon\leftharpoonup z)*h^* = \langle h^*_{(1)},z\rangle h^*_{(2)}=h^*\leftharpoonup z \qquad (69)$$

Proof. We only prove (68). For all $h\in H$, we have

$$\begin{aligned}\langle(z\rightharpoonup\varepsilon)*h^*,h\rangle &= \langle\varepsilon,h_{(1)}z\rangle\langle h^*,h_{(2)}\rangle=\langle\varepsilon,h_{(1)}1_{(1)}z\rangle\langle h^*,h_{(2)}1_{(2)}\rangle\\ &= \langle\varepsilon*h^*,hz\rangle=\langle h^*,hz\rangle=\langle z\rightharpoonup h^*,h\rangle=\langle h^*_{(2)},z\rangle\langle h^*_{(1)},h\rangle.\end{aligned}$$

□

Lemma 4.2. *Let H be a weak Hopf algebra with bijective antipode. For all $y\in H_s$, $z\in H_t$, we have*

$$S^{-1}(z)\rightharpoonup\varepsilon=z\rightharpoonup\varepsilon \text{ and } \varepsilon\leftharpoonup y=\varepsilon\leftharpoonup S^{-1}(y). \qquad (70)$$

Proof. For all $h\in H$, we have

$$\begin{aligned}\langle S^{-1}(z)\rightharpoonup\varepsilon,h\rangle &= \varepsilon(hS^{-1}(z))\overset{(2)}{=}\varepsilon(h1_{(1)})\varepsilon(1_{(2)}S^{-1}(z))\overset{1.5}{=}\varepsilon(h1_{(1)})\varepsilon(zS(1_{(2)}))\\ &\overset{(21)}{=} \varepsilon(h_{(1)})\varepsilon(z\varepsilon_s(h_{(2)}))\overset{(8)}{=}\varepsilon(\varepsilon_s(h)z)\overset{(5)}{=}\varepsilon(hz)=\langle z\rightharpoonup\varepsilon,h\rangle.\end{aligned}$$

The second statement can be proved in a similar way. □

Proposition 4.3. *Let H be a finitely generated projective weak Hopf algebra. Then* $\mathrm{Ker}\,(p)$ *is the k-linear span J of elements of the form*

$$A=hz\bowtie h^*-h\bowtie(z\rightharpoonup\varepsilon)*h^* \text{ and } B=hy\bowtie h^*-h\bowtie(\varepsilon\leftharpoonup y)*h^*,$$

where $h\in H$, $h^*\in H^*$, $y\in H_s$ and $z\in H_t$.

Proof. $A \in \mathrm{Ker}\,(p)$ since

$$\begin{aligned}(1\bowtie\varepsilon)(hz\bowtie h^*)&\overset{(65)}{=}h_{(2)}1_{(2)}\bowtie\varepsilon_{(2)}*h^*\langle\varepsilon_{(1)},h_{(3)}1_{(3)}\rangle\langle\varepsilon_{(3)},S^{-1}(h_{(1)}1_{(1)}z)\rangle\\
&= h_{(2)}\bowtie\varepsilon_{(2)}*h^*\langle\varepsilon_{(1)},h_{(3)}\rangle\langle\varepsilon_{(3)},S^{-1}(z)\rangle\langle\varepsilon_{(4)},S^{-1}(h_{(1)})\rangle\\
&\overset{(66)}{=} h_{(2)}\bowtie(\varepsilon_{(2)}*(S^{-1}(z)\rightharpoonup\varepsilon)*h^*)\langle\varepsilon_{(1)},h_{(3)}\rangle\langle\varepsilon_{(3)},S^{-1}(h_{(1)})\rangle\\
&\overset{(65)}{=} (1\bowtie\varepsilon)\big(h\bowtie((S^{-1}(z)\rightharpoonup\varepsilon)*h^*)\big)\overset{(70)}{=}(1\bowtie\varepsilon)(h\bowtie(z\rightharpoonup\varepsilon)*h^*).\end{aligned}$$

In a similar way, $B \in \mathrm{Ker}\,(p)$:

$$\begin{aligned}(1\bowtie\varepsilon)(hy\bowtie h^*)&\overset{(65)}{=}(h_{(2)}1_{(2)}\bowtie\varepsilon^*_{(2)}*h^*)\langle\varepsilon_{(1)},h_{(3)}1_{(3)}y\rangle\langle\varepsilon_{(3)},S^{-1}(h_{(1)}1_{(1)})\rangle\\
&= (h_{(2)}\bowtie\varepsilon^*_{(3)}*h^*)\langle\varepsilon_{(1)},h_{(3)}\rangle\langle\varepsilon_{(2)},y\rangle\langle\varepsilon_{(4)},S^{-1}(h_{(1)})\rangle\\
&\overset{(67)}{=} (h_{(2)}\bowtie(\varepsilon_{(2)}\leftharpoonup y)*h^*)\langle\varepsilon_{(1)},h_{(3)}\rangle\langle\varepsilon_{(3)},S^{-1}(h_{(1)})\rangle\\
&\overset{(65)}{=} (1\bowtie\varepsilon)(h\bowtie(\varepsilon\leftharpoonup y)*h^*).\end{aligned}$$

This shows that $J \subset \mathrm{Ker}\,(p)$. We now compute for all $h \in H$ and $h^* \in H^*$ that

$$(h\bowtie h^*)(1\bowtie\varepsilon)\overset{(65)}{=}(h1_{(2)}1_{(1')}\bowtie h^*_{(2)})\langle h^*_{(1)},1_{(2')}\rangle\langle h^*_{(3)},S^{-1}(1_{(1)})\rangle,$$

and

$$\begin{aligned}&\Big(h\bowtie(S^{-1}(1_{(2)})\rightharpoonup\varepsilon)*(\varepsilon\leftharpoonup 1_{(1')})*h^*_{(2)}\Big)\langle h^*_{(1)},1_{(2')}\rangle\langle h^*_{(3)},S^{-1}(1_{(1)})\rangle\\
&\overset{(62)}{=} \Big(h\bowtie\varepsilon_{(1)}*\varepsilon_{(2')}*h^*_{(2)}\Big)\langle\varepsilon_{(2)},S^{-1}(1_{(2)})\rangle\langle\varepsilon_{(1')},1_{(1')}\rangle\\
&\qquad\langle h^*_{(1)},1_{(2')}\rangle\langle h^*_{(3)},S^{-1}(1_{(1)})\rangle\\
&= \Big(h\bowtie\varepsilon_{(1)}*\varepsilon_{(2')}*h^*_{(2)}\Big)\langle\varepsilon_{(1')}*h^*_{(1)},1\rangle\langle\varepsilon_{(2)}*h^*_{(3)},S^{-1}(1)\rangle\\
&= \Big(h\bowtie\varepsilon_{(1)}*h^*_{(1)}\Big)\langle\varepsilon_{(2)}*h^*_{(2)},1\rangle = h\bowtie(\varepsilon*h^*)=h\bowtie h^*.\end{aligned}$$

Observing that

$$\begin{aligned}&hzy\bowtie h^*-h\bowtie((S^{-1}(z)\rightharpoonup\varepsilon)*(\varepsilon\leftharpoonup y)*h^*)\\
&\quad= hzy\bowtie h^*-hz\bowtie(\varepsilon\leftharpoonup y)*h^*)\\
&\quad+ hz\bowtie(\varepsilon\leftharpoonup y)*h^*)-h\bowtie((S^{-1}(z)\rightharpoonup\varepsilon)*(\varepsilon\leftharpoonup y)*h^*)\in J,\end{aligned}$$

it follows that $(h \bowtie h^*)(1 \bowtie \varepsilon) - (h \bowtie h^*) \in J$, for all $h \in H$ and $h^* \in H^*$. If $x \in \mathrm{Ker}\,(p)$, then $x(1 \bowtie \varepsilon) = 0$, and $x = x - x(1 \bowtie \varepsilon) \in J$. We conclude that $\mathrm{Ker}\,(p) \subset J$, finishing our proof. □

We now recall the following results from [17]. On $H^*\otimes H$, there exists an associative multiplication

$$\begin{aligned}(h^*\otimes h)(k^*\otimes k) &= k^*_{(2)}h^*\otimes h_{(2)}k\langle S(h_{(1)}),k^*_{(1)}\rangle\langle h_{(3)},k^*_{(3)}\rangle\\
&= (h_{(3)}\rightharpoonup k^*\leftharpoonup S(h_{(1)}))*h^*\otimes h_{(2)}k.\end{aligned}$$

The k-module I generated by elements of the form

$$A' = h^*\otimes hz-(\varepsilon\leftharpoonup z)h^*\otimes h \text{ and } B' = h^*\otimes yh-(y\rightharpoonup\varepsilon)h^*\otimes h$$

is a two-sided ideal of $H^*\otimes H$. The quotient $D'(H) = (H^*\otimes H)/I$ is an algebra with unit element $\varepsilon\otimes 1$. It is a weak Hopf algebra, with the following comultiplication,

counit and antipode:

$$\Delta[h^*\otimes h] = [h^*_{(1)}\otimes h_{(1)}]\otimes [h^*_{(2)}\otimes h_{(2)}] \tag{71}$$

$$\varepsilon[h^*\otimes h] = \langle h^*, \varepsilon_t(h)\rangle \tag{72}$$

$$S[h^*\otimes h] = [S^{-1}(h^*_{(2)})\otimes S(h_{(2)})]\langle h^*_{(1)}, h_{(1)}\rangle\langle h^*_{(3)}, S(h_{(3)})\rangle \tag{73}$$

Proposition 4.4. *The k-linear isomorphism*

$$f:\ H\bowtie H^*\to H^*\otimes H,\ f(h\bowtie h^*) = h^*\otimes S^{-1}(h)$$

is anti-multiplicative, and induces an algebra isomorphism $f:\ D(H)\to D'(H)^{\rm op}$.

Proof. Let us first prove that f reverses the multiplication. Indeed,

$$\begin{aligned} f(k\bowtie k^*)f(h\bowtie h^*) &= (k^*\otimes S^{-1}(k))(h^*\otimes S^{-1}(h))\\ &= (S^{-1}(k_{(1)})\rightharpoonup h^*\leftharpoonup k_{(3)}) * k^*\otimes S^{-1}(k_{(2)})S^{-1}(h)\\ &= f((h\bowtie h^*)(k\bowtie k^*)). \end{aligned}$$

Using Lemma 4.2, we easily compute that $f(J) = I$, and the result follows. □

Let us now define a comultiplication, counit and antipode on $D(H)$, in such a way that $f:\ D(H)\to D'(H)$ is an isomorphism of Hopf algebras. Obviously, the comultiplication is given by the formula

$$\Delta[h\bowtie h^*] = [h_{(2)}\bowtie h^*_{(1)}]\otimes [h_{(1)}\bowtie h^*_{(2)}]. \tag{74}$$

The counit is computed as follows:

$$\varepsilon[h\bowtie h^*] = \varepsilon[h^*\otimes S^{-1}(h)] \overset{(72)}{=} \langle h^*, \varepsilon_t(S^{-1}(h))\rangle \overset{(15)}{=} \langle h^*, 1_{(2)}\rangle\langle \varepsilon, h1_{(1)}\rangle. \tag{75}$$

Since the antipode of H is the inverse of the antipode of $H^{\rm op}$, the antipode of $D'(H)$ is transported to the inverse of the antipode of $D(H)$. We find

$$\begin{aligned} S^{-1}[h\bowtie h^*] &= (f^{-1}\circ S\circ f)[h\bowtie h^*] = f^{-1}(S[h^*\otimes S^{-1}(h)])\\ &= f^{-1}[S^{-1}(h^*_{(2)})\otimes h_{(2)}]\langle h^*_{(1)}, S^{-1}(h_{(3)})\rangle\langle h^*_{(3)}, h_{(1)}\rangle\\ &= [S(h_{(2)})\bowtie S^{-1}(h^*_{(2)})]\langle h^*_{(1)}, S^{-1}(h_{(3)})\rangle\langle h^*_{(3)}, h_{(1)}\rangle \end{aligned} \tag{76}$$

The antipode S is then given by the formula

$$S[h\bowtie h^*] = [S^{-1}(h_{(2)})\bowtie S(h^*_{(2)})]\langle h^*_{(1)}, S^{-1}(h_{(3)})\rangle\langle h^*_{(3)}, h_{(1)}\rangle \tag{77}$$

Indeed,

$$\begin{aligned} &S(S^{-1}[h\bowtie h^*])\\ &\quad= [h_{(3)}\bowtie h^*_{(3)}]\langle h^*_{(1)}, S^{-1}(h_{(5)})\rangle\langle h^*_{(2)}, h_{(4)}\rangle\langle h^*_{(5)}, h_{(1)}\rangle\langle h^*_{(4)}, S^{-1}(h_{(2)})\rangle\\ &\quad= [h_{(3)}\bowtie h^*_{(2)}]\langle h^*_{(1)}, S^{-1}(h_{(5)})h_{(4)}\rangle\langle h^*_{(2)}, S^{-1}(h_{(2)})h_{(1)}\rangle\\ &\quad= [h_{(2)}\bowtie h^*_{(2)}]\langle h^*_{(1)}, \varepsilon_t(S^{-1}(h_{(3)}))\rangle\langle h^*_{(3)}, \varepsilon_t(S^{-1}(h_{(1)}))\rangle\\ &\quad= \varepsilon([h\bowtie h^*]_{(1)})[h\bowtie h^*]_{(2)}\varepsilon([h\bowtie h^*]_{(3)}) = [h\bowtie h^*]. \end{aligned}$$

Similar arguments show that $S^{-1}(S[h\bowtie h^*]) = [h\bowtie h^*]$.

Proposition 4.5. *Let H be a weak Hopf algebra with bijective antipode, which is finitely generated and projective as a k-module. Then $D(H)$ is a weak Hopf algebra, with comultiplication, counit and antipode given by the formulas (74,75,76). As a weak Hopf algebra, $D(H)$ is isomorphic to $D'(H)^{\rm op}$.*

Proposition 4.6. *Let H be a weak Hopf algebra with bijective antipode, which is finitely generated and projective as a k-module. The functor*

$$F:\ {}_H\mathcal{YD}^H \to {}_{D(H)}\overline{\mathcal{M}},\ F(M)=M,$$

with

$$(h\bowtie h^*)m=\langle h^*,m_{[1]}\rangle hm_{[0]},$$

for all $h\in H$, $h^*\in H^*$ and $m\in M$ is an isomorphism of monoidal categories.

Proof. We already know (see (37)) that F is an isomorphism of categories, so we only have to show that F preserves the product. Take $M,N\in {}_H\mathcal{YD}^H$. The right H-coaction on $M\otimes_t N$ is given by the formula (use (48) and (52)):

$$\rho(1_{(1)}m\otimes 1_{(2)}n)=m_{[0]}\otimes n_{[0]}\otimes n_{[1]}m_{[1]},$$

hence the left $D(H)$-action on $F(M\otimes_t N)$ is the following

$$[h\bowtie h^*](1_{(1)}m\otimes 1_{(2)}n)=\langle h^*,n_{[1]}m_{[1]}\rangle h_{(1)}m_{[0]}\otimes h_{(2)}n_{[0]}. \tag{78}$$

We now compute

$$F(N)\otimes_t F(M)=\{[1\bowtie\varepsilon]X\mid X\in F(N)\otimes F(M)\}.$$

Observe that

$$\begin{aligned}[1\bowtie\varepsilon]_{(1)}n\otimes[1\bowtie\varepsilon]_{(2)}m&=\langle\varepsilon_{(1)},n_{[1]}\rangle 1_{(2)}n_{[0]}\otimes\langle\varepsilon_{(2)},m_{[1]}\rangle 1_{(1)}m_{[0]}\\ &=\langle\varepsilon,n_{[1]}m_{[1]}\rangle 1_{(2)}n_{[0]}\otimes 1_{(1)}m_{[0]}.\end{aligned}$$

We claim that the switch map $\tau:\ M\otimes N\to N\otimes M$ induces an isomorphism $\tau:\ F(M\otimes_t N)\to F(N)\otimes_t F(M)$ of k-modules. Indeed, take $1_{(1)}m\otimes 1_{(2)}n\in M\otimes_t N$. Since $M\otimes_t N$ is a Yetter-Drinfeld module, we have that $\varepsilon(n_{[1]}m_{[1]})m_{[0]}\otimes n_{[0]}=1_{(1)}m\otimes 1_{(2)}n$, and

$$\begin{aligned}\tau(1_{(1)}m\otimes 1_{(2)}n)&=1_{(2)}n\otimes 1_{(1)}m=1_{(2')}1_{(2)}n\otimes 1_{(1')}1_{(1)}m\\ &=\varepsilon(n_{[1]}m_{[1]})1_{(2)}n_{[0]}\otimes 1_{(1)}m_{[0]}\\ &=[1\bowtie\varepsilon]_{(1)}n\otimes[1\bowtie\varepsilon]_{(2)}m\in F(N)\otimes_t F(M).\end{aligned}$$

Conversely,

$$\tau([1\bowtie\varepsilon]_{(1)}n\otimes[1\bowtie\varepsilon]_{(2)}m)=\varepsilon(n_{[1]}m_{[1]})1_{(1)}m_{[0]}\otimes 1_{(2)}n_{[0]}\in F(M\otimes_t N).$$

Let us now show that τ is left $D(H)$-linear. To this end, we compute the left $D(H)$-action on $F(N)\otimes_t F(M)$.

$$\begin{aligned}[h\bowtie h^*]\tau(1_{(1)}m\otimes 1_{(2)}n)&=[h\bowtie h^*](1_{(2)}n\otimes 1_{(1)}m)\\ &=[h_{(2)}\bowtie h^*_{(1)}](1_{(2)}n)\otimes[h_{(1)}\bowtie h^*_{(2)}](1_{(1)}m)\\ &\overset{(55)}{=}\langle h^*_{(1)},n_{[1]}S^{-1}(1_{(2)})\rangle h_{(2)}n_{[0]}\otimes\langle h^*_{(2)},1_{(1)}m_{[1]}\rangle h_{(1)}m_{[0]}\\ &=\langle h^*,n_{[1]}S^{-1}(1_{(2)})1_{(1)}m_{[1]}\rangle h_{(2)}n_{[0]}\otimes h_{(1)}m_{[0]}\\ &\overset{(78)}{=}\tau\big([h\bowtie h^*](1_{(1)}m\otimes 1_{(2)}n)\big)\end{aligned}$$

It also follows that $F(H_t)$ is a unit object in ${}_{D(H)}\overline{\mathcal{M}}$. Since the unit object in a monoidal category is unique up to automorphism, we concluce that the target space of $D(H)_t$ is isomorphic to H_t. This can also be seen as follows: in [17], it is shown that $D'(H)_t=[\varepsilon\otimes H_t]\cong H_t$. Since the target spaces of a weak Hopf algebra and its opposite coincide, it follows that $D(H)_t\cong H_t$. □

5. Duality

Let H be a weak Hopf algebra with bijective antipode, and ${}_H\mathrm{Rep}$ the category of left H-modules M which are finitely generated projective as a k-module. Let $M \in {}_H\mathrm{Rep}$, and let $\{(n_i, n_i^*) \mid i = 1, \cdots n\}$ be a finite dual basis of M. From [17], we recall the following result. We refer to [12] for the definition of duality in a monoidal category.

Proposition 5.1. *The category ${}_H\mathrm{Rep}$ has left duality. The left dual of $M \in {}_H\mathrm{Rep}$ is $M^* = \mathrm{Hom}(M, k)$ with left H-action defined by*

$$\langle h \cdot m^*, m\rangle = \langle m^*, S(h)m\rangle, \tag{79}$$

for all $h \in H$, $m \in M$ and $m^* \in M^*$. The evaluation map $\mathrm{ev}_M :\ M^* \otimes_t M \to H_t$ and the coevaluation map $\mathrm{coev}_M :\ H_t \to M \otimes_t M^*$ are defined as follows:

$$\mathrm{ev}_M(1_{(1)} \cdot m^* \otimes 1_{(2)}m) = \langle m^*, 1_{(1)}m\rangle 1_{(2)};$$

$$\mathrm{coev}_M(z) = z \cdot (\textstyle\sum_i n_i \otimes n_i^*).$$

Let M be a finitely generated projective left H-comodule. Then M^* is also a left H-comodule, with left H-coaction $\lambda :\ M^* \to H \otimes M^*$ given by

$$\lambda(m^*) = \textstyle\sum_i \langle m^*, n_{i[0]}\rangle S^{-1}(n_{i[-1]}) \otimes n_i^*.$$

The definition of λ can also be stated as follows: $\lambda(m^*) = m^*_{[-1]} \otimes m^*_{[0]}$ if and only if

$$\langle m^*_{[0]}, m\rangle S(m^*_{[-1]}) = \langle m^*, m_{[0]}\rangle m_{[-1]}, \tag{80}$$

for all $m \in M$.

Proposition 5.2. *Let M be a finitely generated projective left-left Yetter-Drinfeld module over the weak Hopf algebra H. Then M^* with H-action and H-coaction given by (79) and (80) is also a left-left Yetter-Drinfeld module.*

Proof. We have to show that

$$\lambda(h \cdot m^*) = \textstyle\sum_i \langle m^*, S(h)n_{i[0]}\rangle S^{-1}(n_{i[-1]}) \otimes n_i^*$$

equals

$$h_{(1)}m^*_{[-1]}S(h_{(3)}) \otimes h_{[2]}m^*_{[-1]} = \textstyle\sum_i \langle m^*, n_{i[0]}\rangle h_{(1)}S^{-1}(n_{i[-1]})S(h_{(3)}) \otimes (h_{(2)} \cdot n_i^*).$$

It suffices to show that both terms coincide after we evaluate the second tensor factor at an arbitrary $m \in M$.

$$\begin{aligned}
&\textstyle\sum_i \langle m^*, n_{i[0]}\rangle h_{(1)}S^{-1}(n_{i[-1]})S(h_{(3)})\langle n_i^*, S(h_{(2)})m\rangle \\
&\quad = \quad \langle m^*, (S(h_{(2)})m)_{[0]}\rangle h_{(1)}S^{-1}\Big((S(h_{(2)})m)_{[-1]}\Big)S(h_{(3)}) \\
&\quad \overset{(41)}{=} \quad \langle m^*, S(h_{(3)})m_{[0]}\rangle h_{(1)}S^{-1}\Big(S(h_{(4)})m_{[-1]}S^2(h_{(2)})\Big)S(h_{(5)}) \\
&\quad = \quad \langle m^*, S(h_{(3)})m_{[0]}\rangle h_{(1)}S(h_{(2)})S^{-1}(m_{[-1]})h_{(4)}S(h_{(5)}) \\
&\quad = \quad \langle m^*, S(h_{(2)})m_{[0]}\rangle \varepsilon_t(h_{(1)})S^{-1}(m_{[-1]})\varepsilon_t(h_{(3)})
\end{aligned}$$

$$\begin{aligned}
&\overset{(21)}{=} && \langle m^*, S(1_{(2)}h_{(1)})m_{[0]}\rangle S(1_{(1)})S^{-1}(m_{[-1]})\varepsilon_t(h_{(2)})\\
&= && \langle m^*, S(h_{(1)})1_{(1)}m_{[0]}\rangle 1_{(2)}S^{-1}(m_{[-1]})\varepsilon_t(h_{(2)})\\
&\overset{(3,44)}{=} && \langle m^*, S(1_{(1)}h)m_{[0]}\rangle S^{-1}(m_{[-1]})1_{(2)}\\
&= && \langle m^*, S(h)S(1_{(1)})m_{[0]}\rangle S^{-1}(S(1_{(2)})m_{[-1]})\\
&= && \langle m^*, S(h)1_{(2)}m_{[0]}\rangle S^{-1}(1_{(1)}m_{[-1]})\\
&\overset{(38)}{=} && \langle m^*, S(h)m_{[0]}\rangle S^{-1}(m_{[-1]})\\
&= && \textstyle\sum_i \langle m^*, S(h)n_{i[0]}\rangle S^{-1}(n^*_{i[-1]})\langle n_i^*, m\rangle
\end{aligned}$$

□

Proposition 5.3. *The category of finitely generated projective left-left Yetter-Drinfeld modules has left duality.*

Proof. In view of the previous results, it suffices to show that the evaluation map ev_M and the coevaluation map coev_M are left H-colinear, for every finitely generated projective left-left Yetter-Drinfeld module M. Let us first show that ev_M is left H-colinear.

$$\begin{aligned}
&(H\otimes \mathrm{ev}_M)(\lambda(1_{(1)}\cdot m^*\otimes 1_{(2)}m))\\
&\quad= m^*_{[-1]}m_{[-1]}\otimes \langle m^*_{[0]}, 1_{(1)}m_{[0]}\rangle 1_{(2)}\\
&\quad\overset{(80)}{=} \langle m^*, (1_{(1)}m_{[0]})_{[0]}\rangle S^{-1}((1_{(1)}m_{[0]})_{[-1]})m_{[-1]}\otimes 1_{(2)}\\
&\quad\overset{(43)}{=} \langle m^*, m_{[0]}\rangle 1_{(1)}S^{-1}(m_{[-1]})m_{[-2]}\otimes 1_{(2)}\\
&\quad= \langle m^*, m_{[0]}\rangle 1_{(1)}\varepsilon_t(S^{-1}(m_{[-1]}))\otimes 1_{(2)}\\
&\quad\overset{(44)}{=} \langle m^*, 1_{(1')}m_{[0]}\rangle 1_{(1)}\varepsilon_t(1_{(2')}S^{-1}(m_{[-1]}))\otimes 1_{(2)}\\
&\quad\overset{(10)}{=} \langle m^*, 1_{(1')}m_{[0]}\rangle 1_{(1)}1_{(2')}\varepsilon_t(S^{-1}(m_{[-1]}))\otimes 1_{(2)}\\
&\quad\overset{(25)}{=} \langle m^*, 1_{(1')}S^{-1}(\varepsilon_t(S^{-1}(m_{[-1]})))m_{[0]}\rangle 1_{(1)}1_{(2')}\otimes 1_{(2)}\\
&\quad\overset{(22)}{=} \langle m^*, 1_{(1')}\varepsilon_s(S^{-2}(m_{[-1]}))m_{[0]}\rangle 1_{(1)}1_{(2')}\otimes 1_{(2)}\\
&\quad\overset{(1,45)}{=} \langle m^*, 1_{(1)}m\rangle 1_{(2)}\otimes 1_{(3)} \overset{(49)}{=} \lambda(\langle m^*, 1_{(1)}m\rangle 1_{(2)})\\
&\quad= \lambda(\mathrm{ev}_M(1_{(1)}\cdot m^*\otimes 1_{(2)}m)).
\end{aligned}$$

To prove that coev_M is left H-colinear, we have to show that, for all $z\in H_t$,

$$\begin{aligned}
\lambda(\mathrm{coev}_M(z)) &= \textstyle\sum_i \lambda(1_{(1)}zn_i\otimes 1_{(2)}\cdot n_i^*)\\
&= \textstyle\sum_i (1_{(1)}zn_i)_{[-1]}(1_{(2)}\cdot n_i^*)_{[-1]}\otimes (1_{(1)}zn_i)_{[0]}\otimes (1_{(2)}\cdot n_i^*)_{[0]}
\end{aligned}$$

equals

$$(H\otimes \mathrm{coev}_M)(\lambda(z)) = (H\otimes \mathrm{coev}_M)(1_{(1)}z\otimes 1_{(2)}) = \textstyle\sum_i 1_{(1)}z\otimes 1_{(2)}n_i\otimes 1_{(3)}\cdot n_i^*.$$

It suffices to show that both terms coincide after we evaluate the third tensor factor at an arbitrary $m\in M$. Indeed

$$\begin{aligned}
&\textstyle\sum_i (1_{(1)}zn_i)_{[-1]}(1_{(2)}\cdot n_i^*)_{[-1]}\otimes (1_{(1)}zn_i)_{[0]}\langle (1_{(2)}\cdot n_i^*)_{[0]}, m\rangle\\
&\quad\overset{(80)}{=} \textstyle\sum_i (1_{(1)}zn_i)_{[-1]}\langle 1_{(2)}\cdot n_i^*, m_{[0]}\rangle S^{-1}(m_{[-1]})\otimes (1_{(1)}zn_i)_{[0]}\\
&\quad= \textstyle\sum_i (1_{(1)}zn_i)_{[-1]}\langle n_i^*, S(1_{(2)})m_{[0]}\rangle S^{-1}(m_{[-1]})\otimes (1_{(1)}zn_i)_{[0]}\\
&\quad= (1_{(1)}zS(1_{(2)})m_{[0]})_{[-1]}S^{-1}(m_{[-1]})\otimes (1_{(1)}zS(1_{(2)})m_{[0]})_{[0]}
\end{aligned}$$

$$\begin{aligned}
&\overset{(6,41)}{=} && 1_{(1)}zm_{[-1]}S(1_{(3)})S^{-1}(m_{[-2]})\otimes 1_{(2)}m_{[0]}\\
&= && 1_{(1)}zm_{[-1]}S(1_{(2')})S^{-1}(m_{[-2]})\otimes 1_{(2)}1_{(1')}m_{[0]}\\
&\overset{(42)}{=} && 1_{(1)}zm_{[-1]}S^{-1}(m_{[-2]})\otimes 1_{(2)}m_{[0]}\\
&= && 1_{(1)}zS^{-1}(\varepsilon_t(m_{[-1]}))\otimes 1_{(2)}m_{[0]}\\
&\overset{(25)}{=} && 1_{(1)}z\otimes 1_{(2)}\varepsilon_t(m_{[-1]})m_{[0]}\overset{(40)}{=}1_{(1)}z\otimes 1_{(2)}m\\
&= && 1_{(1)}z\otimes 1_{(2)}S(1_{(3)})m=\textstyle\sum_i 1_{(1)}z\otimes 1_{(2)}n_i\langle n_i^*,S(1_{(3)})m\rangle\\
&= && \textstyle\sum_i 1_{(1)}z\otimes 1_{(2)}n_i\langle 1_{(3)}\cdot n_i^*,m\rangle,
\end{aligned}$$

as needed. □

6. Appendix. Weak bialgebras and bialgebroids

In [21], Yetter-Drinfeld modules over a $\times_R$-bialgebra (see [24]) are introduced, and it is shown that the weak center of the category of left modules is isomorphic to the category of Yetter-Drinfeld modules. The notion of $\times_R$-bialgebra is equivalent to the notion of R-bialgebroid, we refer to [3] for a detailed discussion. So we can consider Yetter-Drinfeld modules over bialgebroids.

A weak bialgebra H can be viewed as a bialgebroid over the target space H_t; this was shown in [9] in the weak Hopf algebra case, and generalized to the weak bialgebra case in [22]. The aim of this Section is to make clear that Yetter-Drinfeld modules over H considered as a weak bialgebra coincide with Yetter-Drinfeld modules over H-considered as a bialgebroid.

To this end, we first recall the definition of a bialgebroid, as introduced by Lu [14]. Let k be a commutative ring, and R a k-algebra. An $R\otimes R^{\rm op}$-ring is a pair (H,i), with H a k-algebra and $i:\ R\otimes R^{\rm op}\to H$. Giving i is equivalent to giving algebra maps $s_H:\ R\to H$ and $t_H:\ R\to H^{\rm op}$ satisfying $s_H(a)t_H(b)=t_H(b)s_H(a)$, for all $a,b\in R$. We then have that $i(a\otimes b)=s_H(a)t_H(b)$. Restriction of scalars makes H into a left $R\otimes R^{\rm op}$-module, and an R-bimodule:

$$a\cdot h\cdot b=s_H(a)t_H(b)h.$$

Consider

$$\begin{aligned}H\times_R H=\{&\sum_i h_i\otimes_R k_i\in H\otimes_R H\\ &\mid\ \sum_i h_it_H(a)\otimes_R k_i=\sum_i h_i\otimes_R k_is_H(a),\ \text{for all } a\in R\}\end{aligned}$$

It is easy to show that $H\times_R H$ is a k-subalgebra of $H\otimes_R H$.

Recall that an R-coring is a triple $(H,\tilde{\Delta},\tilde{\varepsilon})$, with H an R-bimodule and $\tilde{\Delta}:\ H\to H\otimes_R H$ and $\tilde{\varepsilon}:\ H\to R$ R-bimodule maps satisfying the usual coassociativity and counit properties; we refer to [4] for a detailed discussion of corings.

Definition 6.1. [14] A left R-bialgebroid is a fivetuple $(H,s_H,t_H,\tilde{\Delta},\tilde{\varepsilon})$ satisfying the following conditions.

(1) $(H,\tilde{\Delta},\tilde{\varepsilon})$ is an R-coring;
(2) $(H,m\circ(s_H\otimes t_H)=i)$ is an $R\otimes R^{\rm op}$-ring;
(3) $\mathrm{Im}\,(\tilde{\Delta})\subset H\times_R H$;
(4) $\tilde{\Delta}:\ H\to H\times_R H$ is an algebra map, $\tilde{\varepsilon}(1_H)=1_R$ and

$$\tilde{\varepsilon}(gh)=\tilde{\varepsilon}(gs_H(\tilde{\varepsilon}(h)))=\tilde{\varepsilon}(gt_H(\tilde{\varepsilon}(h))),$$

for all $g, h \in H$.

Take two left H-modules M and N; then M and N are R-bimodules, by restriction of scalars. $M \otimes_R N$ is a left H-module, with

$$h \cdot (m \otimes_R n) = h_{(1)} m \otimes_R h_{(2)} n.$$

Also R is a left H-module, with

$$h \cdot r = \tilde{\varepsilon}(h s_H(r)) = \tilde{\varepsilon}(h t_H(r)).$$

$({}_H\mathcal{M}, \otimes_R, R)$ is a monoidal category, and the restriction of scalars functor ${}_H\mathcal{M} \to {}_R\mathcal{M}_R$ is strictly monoidal; this can be used to reformulate the definition of a bialgebroid (see [3, 20, 23]).
In [21, Sec. 4], left-left Yetter-Drinfeld modules over H are introduced, and it is shown that $\mathcal{W}_l({}_H\mathcal{M})$ is isomorphic to the category of Yetter-Drinfeld modules. According to [21], a left-left Yetter-Drinfeld H-module is a left comodule M over the coring H, together with a left H-action on M such that the underlying left R-actions coincide, and such that

$$h_{(1)} m_{[-1]} \otimes_R h_{(2)} \cdot m_{[0]} = (h_{(1)} \cdot m)_{[-1]} h_{(2)} \otimes_R (h_{(1)} \cdot m)_{[0]} \tag{81}$$

holds in $H \otimes_R M$, for all $h \in H$ and $m \in M$.

Let H be a weak bialgebra, and consider the maps

$$\begin{aligned}
&s_H : \ H_t \xrightarrow{\subset} H;\\
&t_H = \overline{\varepsilon}_{s|H_t} : \ H_t \to H_s \subset H;\\
&\tilde{\Delta} = \mathrm{can} \circ \Delta : \ H \to H \otimes H \xrightarrow{\mathrm{can}} H \otimes_{H_t} H;\\
&\tilde{\varepsilon} = \varepsilon_t : \ H \to H_t.
\end{aligned}$$

Then $(H, s_H, t_H, \tilde{\Delta}, \tilde{\varepsilon})$ is a left H_t-bialgebroid. The fact that $\mathrm{Im}\,(\tilde{\Delta}) \subset H \times_{H_t} H$ follows from the separability of H_t as a k-algebra (cf. Proposition 1.3).
We have seen in Section 1.1 that, for any two left H-modules M and N, we have an isomorphism $\overline{\pi} : \ M \otimes_{H_t} N \to M \otimes_t N$. This entails that the monoidal categories $({}_H\mathcal{M}, \otimes_t, H_t)$ and $({}_H\mathcal{M}, \otimes_{H_t}, H_t)$ are isomorphic, and a fortiori, their weak left centers are isomorphic categories. Consequently, the two corresponding categories of Yetter-Drinfeld modules are isomorphic. This can also be seen directly, comparing the definitions in Section 2 and (81).

Acknowledgment

We thank Tomasz Brzeziński and the referee for their useful comments, and Adriana Nenciu for sending us her paper [16].

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